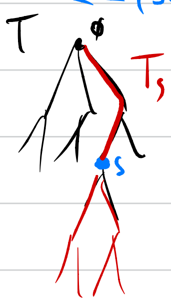
Descriptive Set Theory
Lecture 4

Prop. For a pruned tree T on A, $[T]$ is compact if and only if T is finitely-branching.

Proof. $\Rightarrow$. Done.

$\Leftarrow$ (Jenna Zomback). Let $\mathcal{U}$ be an open cover of $[T]$ and suppose towards a contradiction that $\nexists$ finite subcover.



We call $s \in T$ *heavy* if $[T_s]$ cannot be covered by a finite subset of $\mathcal{U}$, where $T_s := \{t \in T : t \subseteq s \text{ or } t \supseteq s\}$. Thus, we know that $\emptyset$ is heavy. Then one of the extensions of $\emptyset$ must be heavy (finite union of finite sets is finite) and we continue, finding a branch $x \in [T]$ with the property that $[T_{x|n}]$ cannot be covered by a finite subset of $\mathcal{U}$ for each n. But $\mathcal{U}$ covers $[T]$ so $\exists\, U \in \mathcal{U}$ with $x \in U$. But then for a large enough n, $[T_{x|n}] \subseteq U$, a contradiction. $\square$

In particular, $2^{\mathbb{N}}$ is compact. Also, $\mathbb{N}^{\mathbb{N}}$ is not compact. In fact, $\mathbb{N}^{\mathbb{N}}$ is not even σ-compact (i.e. ctbl union of compact), unlike $\mathbb{R}^n$.

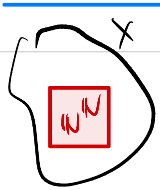
Prop. $\mathbb{N}^{\mathbb{N}}$ is not σ-compact.

Proof. Let $X \subseteq \mathbb{N}^{\mathbb{N}}$ be a σ-compact subset and we aim to find $y \in \mathbb{N}^{\mathbb{N}}$ that is not in X. Note that $X = \bigcup_{n \in \mathbb{N}} [T_n]$, where each T_n is a finitely-branching pruned tree.

For $x, y \in \mathbb{N}^{\mathbb{N}}$, we say that y dominates (resp. eventually dominates) x, if $\forall n\ x(n) \leq y(n)$ (resp. $\forall^\infty n\ x(n) \leq y(n)$)
($\forall^\infty n$ means for all but finitely many, i.e. $\exists N \forall n \geq N$.)
($\exists^\infty n$ means for infinitely many, i.e. $\forall N \exists n \geq N$.)
By finite branching, each $[T_n]$ is dominated by an $x_n \in [T_n]$. By diagonalization, we get a $y \in \mathbb{N}^{\mathbb{N}}$ eventually dominating every x_n, namely, for each $i \in \mathbb{N}$, let $y(i) := \max\{x_0(i), x_1(i), x_2(i), \dots, x_i(i)\}$. Indeed, y eventually dominates every x_7 because $\forall i \geq 7$, $y(i) \geq x_7(i)$. Thus, $y \notin X$. $\square$

The following dichotomy shows that $\mathbb{N}^{\mathbb{N}}$ is *the* non-σ-compact space:

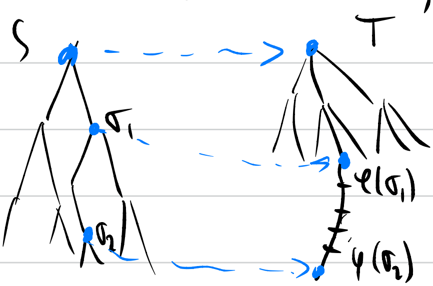
Hurewitz's dichotomy. Every Polish space X is either σ-compact or X has a closed subset homeomorphic to $\mathbb{N}^{\mathbb{N}}$.



Monotone maps and continuous functions.

We will learn how to build continuous functions $[S] \to [T]$, where S, T are trees, and we will see that all continuous functions are built this way.

Let S, T be trees on alphabets A, B, resp. Call a map $\varphi : S \to T$ **monotone** if for all $\sigma_1, \sigma_2 \in S$, $\sigma_1 \subseteq \sigma_2 \implies \varphi(\sigma_1) \subseteq \varphi(\sigma_2)$, and $\varphi(\emptyset) = \emptyset$.



Let $D_\varphi := \{x \in [S] : |\varphi(x|_n)| \to \infty \text{ as } n \to \infty\}$

Thus, φ induces a function

$$\varphi^* : D_\varphi \to [T]$$
$$x \mapsto \bigcup_n \varphi(x|_n)$$

Theorem. (a) For any monotone $\varphi : S \to T$, D_φ is a G_δ subset of $[S]$, and φ^* is continuous.

(b) All continuous maps from G_δ subsets of $[S]$ to $[T]$ arise in this fashion.

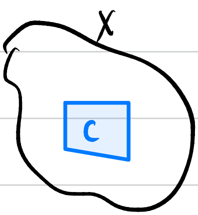
Proof. (b) is left for you.

(a) To see that D_φ is G_δ, we note that $\forall x \in [S]$,

$$x \in D_\varphi \iff \underbrace{\forall \ell \in \mathbb{N} \; \underbrace{\exists n \in \mathbb{N} \; \underbrace{|\varphi(x|_n)| > \ell}_{\text{clopen}}}_{\text{open}}}_{G_\delta}.$$

Note $\{x \in [S] : |\varphi(x|_n)| > \ell\}$ is a clopen set here membership in it depends only on first n coordinates. $\exists n \in \mathbb{N}$ is the same as $\bigcup_{n \in \mathbb{N}}$ and $\forall \ell \in \mathbb{N}$ is the same as $\bigcap_{\ell \in \mathbb{N}}$. □

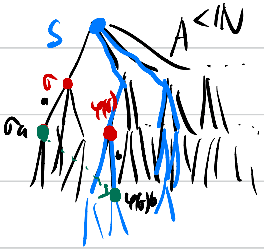
As a quick application, we consider retractions. A closed subset C of a topol. space X is called a retract of X if



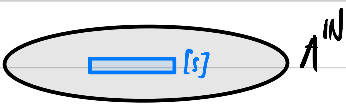
$\exists$ continuous $\pi : X \twoheadrightarrow C$ s.t. $\pi|_C = id_C$. This map π is called a retraction of X onto C.

Cor. For any alphabet A, and closed subset $C \subseteq A^{\mathbb{N}}$ is a retract of $A^{\mathbb{N}}$. In particular, for closed sets $C_1 \subseteq C_2$, C_1 is a retract of C_2.

Proof. Let $C = [S]$ for some pruned tree S on A.



We define $\varphi : A^{<\mathbb{N}} \to S$ as follows: $\varphi(\emptyset) := \emptyset$. Let $\sigma \in A^{<\mathbb{N}}$ s.t. $\varphi(\sigma)$ is defined but $\varphi(\sigma a)$ isn't defined yet. If $\sigma a \in S$, $\varphi(\sigma a) := \sigma a$. Otherwise, using that σ is pruned,



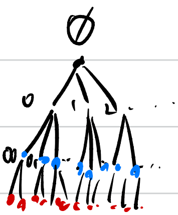
take any $b \in A$ s.t. $\varphi(\sigma) b \in S$. Put $\varphi(\sigma a) := \varphi(\sigma) b$. □

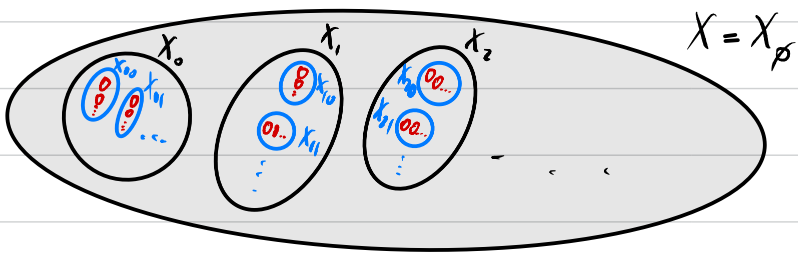
Luzin and Cantor schemes.

A Luzin scheme in a top. space X is a sequence $(X_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ of subsets of X such that

(i) $X_s \subseteq X_t$ if $t \geq s$, $\forall s,t \in \mathbb{N}^{<\mathbb{N}}$.

(ii) $X_{sa} \cap X_{sb} = \emptyset$ $\forall s \in \mathbb{N}^{<\mathbb{N}}$ and $a \neq b \in \mathbb{N}$.





(iii) If X is a metric space with a metric d, then we'd say that the scheme has vanishing diameter if $\forall x \in \mathbb{N}^{\mathbb{N}}$, $\mathrm{diam}_d(X_{x|_n}) \to 0$ as $n \to \infty$.

A scheme with vanishing diameter induces a function $f: D \to X$, where $D := \{x \in \mathbb{N}^{\mathbb{N}} : \bigcap_n X_{x|n} \neq \emptyset\}$.

$x \mapsto$ the unique element of $\bigcap_n X_{x|n}$ (by vanishing diam).

If instead we have $(X_s)_{s\in 2^{<\mathbb{N}}}$, then we call it a Cantor scheme.

Properties of Luzin schemes. Let $(A_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ a Luzin scheme of vanishing diameter in a metric space (X,d), and let $f: \mathcal{D} \to X$ be the induced map.

(a) f is injective and continuous.

(b) If $A_s = \bigcup_{n\in\mathbb{N}} A_{sn}$ $\forall s\in\mathbb{N}^{<\mathbb{N}}$, then f is surjective.

(c) If A_s is open for each $s\in\mathbb{N}^{<\mathbb{N}}$, then f is open.

(d) If d is complete and $\overline{A_{sn}} \subseteq A_s$ $\forall s\in\mathbb{N}^{<\mathbb{N}}$ and $n\in\mathbb{N}$, then $\mathcal{D}$ is closed. In fact, $x\notin\mathcal{D} \iff \exists n\ A_{x|n} = \emptyset$.
In particular, if all $A_s \neq \emptyset$, then $\mathcal{D} = \mathbb{N}^{\mathbb{N}}$.